Isotropic singularity in brane cosmological models

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It is argued that the initial cosmological singularity is isotropic in spatially inhomogeneous brane-world models. This implies that brane cosmology may naturally give rise to a set of initial data that provide the conditions for inflation to subsequently take place, consequently offering a plausible solution to the initial conditions problem in cosmology.

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Cosmology models in which our Universe is a four-dimensional brane embedded in a higher-dimensional spacetime is currently of great interest. In the Brane-world scenario, ordinary matter fields are confined to the brane, while the gravitational field can also propagate in the extra dimensions (i.e., in the 'bulk') [1]. In particular, generalized Randall-Sundrum-type models [2] are relatively simple phenomenological models which are inspired by string theory. These models are self-consistent and simple and allow for an investigation of the essential non-linear gravitational dynamics, particularly in the high-energy regime close to the initial singularity, but captures some of the essential features of the dimensional reduction of eleven-dimensional supergravity introduced by Hořava and Witten [3]. Randall-Sundrum type models have in common a five dimensional space-time (bulk), governed by the Einstein equations, and a four dimensional brane, representing our physical world, on which ordinary matter fields are confined. At low energies gravity is also localized at the brane (even when the extra dimensions is not small) [2]. There are many generalizations of the original Randall-Sundrum scenario which allow for matter with cosmological symmetry on the brane (Friedmann branes) [4] (in this case the bulk is Schwarzschild-Anti de Sitter space-time [5]), and non-empty bulks have also been considered including models allowing a *dilatonic* type scalar field in the bulk [6]. A geometric formulation of the class of Randall-Sundrum-type brane-world models was given in [7]. The dynamical equations on the 3-brane differ from the equations in general relativity (GR) by terms that carry the effects of imbedding and of the free gravitational field in the 5-dimensional bulk (transmitted via the projection $\mathcal{E}_{\mu\nu}$ of the bulk Weyl tensor) [8]. In general, in the 4-dimensional picture the conservation equations do not determine all of the independent components of $\mathcal{E}_{\mu\nu}$ on the brane (and a complete higher-dimensional analysis, including the dynamics in the bulk,is necessary).

The asymptotic dynamical evolution of spatially homogeneous brane-world cosmological models close to the initial singularity, where the energy density of the matter is larger than the brane tension and the behaviour deviates significantly from the classical general-relativistic case, was studied in [9,10]. It was found that for perfect fluid models with a linear barotropic γ-law equation of state an isotropic singularity [11] is a past-attractor in all orthogonal Bianchi models and is a local past-attractor in a class of inhomogeneous brane-world models for all $\gamma > 1$. The early investigations of the initial singularity used only isotropic fluids as a source of matter [12]. However, the study of the behaviour of spatially homogeneous brane-worlds close to the initial singularity in the presence of both local and nonlocal stresses indicates that for physically relevent values of the equation of state parameter there exist two local past attractors for these brane-worlds, one isotropic and one anisotropic (although the anisotropic models are likely unphysical and can be ruled out). In particular, Barrow and Hervik [13] studied a class of Bianchi type I brane-world models with a pure magnetic field and a perfect fluid and found that when $\gamma \geq \frac{4}{3}$ the equilibrium point $\mathcal{F}_b$ is again a local source (past-attractor), but that there exists a second equilibrium point, which corresponds to a new brane-world solution with a non-trivial magnetic field, which is also a local source (and is the only local source when $\gamma < \frac{4}{3}$). This was generalized by [14], in which it was shown that for a class of spatially homogeneous brane-worlds with anisotropic stresses, both local and nonlocal, the brane-worlds could have either an isotropic singularity or an anisotropic singularity for $\gamma > 4/3$.

The *governing field equations* induced on the brane, using the Gauss-Codazzi equations, matching conditions and Z_2 symmetry are given in [7,8]. The energy-momentum tensor of the matter fields can be decomposed covariantly with respect to a chosen 4-velocity (timelike vector) u^μ, in terms of the energy density ρ and isotropic pressure p, and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects orthogonal to u^μ. We are particularly interested in the physical case of a perfect fluid with $\rho = (\gamma - 1)p$, especially with $\gamma \geq 4/3$, where $\gamma = 4/3$ corresponds to radiation and $\gamma = 2$ corresponds to massless scalar fields close to the initial singularity. The dynamical equations on the 3-brane differ from the GR equations [7,8] in that there are nonlocal effects from the free gravitational field in the bulk, transmitted via the projection $\mathcal{E}_{\mu\nu}$ of the

bulk Weyl tensor, and the local quadratic energy-momentum corrections, which are significant at very high energies and particularly close to the initial singularity. Nonlocal effects from the bulk can be irreducibly decomposed in terms of an effective nonlocal energy density, an effective nonlocal anisotropic stress and an effective nonlocal energy flux on the brane, $\mathcal{U}$, $\mathcal{P}_{\mu\nu}$ and $\mathcal{Q}_\mu$, respectively [8].

All of the bulk corrections may be consolidated into an effective total energy density, pressure, anisotropic stress and energy flux, so that the modified Einstein equations take the standard Einstein form with a redefined energy-momentum tensor:

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} + \widetilde{\kappa}^4 S_{\mu\nu} - \mathcal{E}_{\mu\nu} \equiv T^{\rm tot}_{\mu\nu} = \rho^{\rm tot} u_\mu u_\nu + p^{\rm tot} h_{\mu\nu} + \pi^{\rm tot}_{\mu\nu} + 2q^{\rm tot}_{(\mu} u_{\nu)}\,, \tag{1}$$

where $\kappa^2 = 8\pi/M_{\rm p}^2\,,\quad \lambda = 6\kappa^2/\widetilde{\kappa}^4$, and

$$\rho^{\rm tot} = \kappa^2\,\rho + \frac{6\kappa^2}{\lambda}\left[\frac{1}{24}\left(2\rho^2 - 3\pi_{\mu\nu}\pi^{\mu\nu}\right) + \frac{1}{\kappa^4}\mathcal{U}\right] \tag{2}$$

$$p^{\rm tot} = \kappa^2\,p + \frac{6\kappa^2}{\lambda}\left[\frac{1}{24}\left(2\rho^2 + 4\rho p + \pi_{\mu\nu}\pi^{\mu\nu} - 4q_\mu q^\mu\right) + \frac{1}{3}\frac{1}{\kappa^4}\mathcal{U}\right] \tag{3}$$

$$\pi^{\rm tot}_{\mu\nu} = \kappa^2\,\pi_{\mu\nu} + \frac{6\,\kappa^2}{\lambda}\left[\frac{1}{12}\left(-(\rho+3p)\pi_{\mu\nu} - 3\pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^{\alpha} + 3q_{\langle\mu}q_{\nu\rangle}\right) + \frac{1}{\kappa^4}\mathcal{P}_{\mu\nu}\right] \tag{4}$$

$$q^{\rm tot}_\mu = \kappa^2\,q_\mu + \frac{6\,\kappa^2}{\lambda}\left[\frac{1}{12}\left(2\rho q_\mu - 3\pi_{\mu\nu}q^\nu\right) + \frac{1}{\kappa^4}\mathcal{Q}_\mu\right] \tag{5}$$

As a consequence of the form of the bulk energy-momentum tensor and of Z_2 symmetry, it follows [7] that the brane energy-momentum tensor separately satisfies the conservation equations, i.e., $\nabla^\nu T_{\mu\nu} = 0$. Consequently, the Bianchi identities on the brane imply that the projected Weyl tensor obeys the non-local constraint $\nabla^\mu \mathcal{E}_{\mu\nu} = \widetilde{\kappa}^4 \nabla^\mu S_{\mu\nu}$. The evolution of the anisotropic stress part is *not* determined on the brane. The correction terms must be consistently derived from the higher-dimensional equations. The fact that since $\mathcal{P}_{\mu\nu}$ corresponds to gravitational waves in higher-dimensions it is expected that the dynamics will not be affected significantly at early times close to the singularity [17]. Henceforth we shall assume that $\mathcal{P}_{\mu\nu} = 0$. A dynamical argument to support this assumption is given in [15]. No further assumptions on the models are made.

From numerical and dynamical considerations it was concluded that [15] the area expansion rate increases without bound (and hence the Hubble rate $\to \infty$) as logarithmic time $t \to -\infty$, and hence there always exists an *initial singularity*. In addition, the normalized frame variable [16] vanishes as $t \to -\infty$. This allows us to calculate the exponential decay rates close to the initial singularity. The constrained evolution system of equations for general inhomogeneous (G_0) brane world models were given in [26] in terms of the variables $\Omega_b, \mathbf{X}$ (in the separable volume gauge using Hubble-normalized equations), where $\mathbf{X}$ represents the independent variables corresponding to the shear and curvature and additional matter terms, and $\Omega_b \equiv \mu_b/H^2$, where essentially $\mu_b \sim {\rho_b}^2$. The exponential decay rates in the case $\gamma > 4/3$ were calculated by Lim (see the Appendix in [15]). It was found that as $t \to -\infty$ the variables $(\Omega_b - 1), \mathbf{X}$ have decay rates that depend linearly on $\{\mathrm{e}^{(3\gamma-1)t}, \mathrm{e}^{3\gamma t}, \mathrm{e}^{3(\gamma-1)t}\mathrm{e}^{2(3\gamma-4)t}, \mathrm{e}^{2t}, \mathrm{e}^{(3\gamma-4)t}, \mathrm{e}^{2(3\gamma-2)t}\}$. There is also evidence of isotropization (albeit slowly) in the degenerate case $\gamma = 4/3$ (see below). This supports the possibility that in general brane world cosmologies have an isotropic singularity for $\gamma \geq 4/3$.

The evolution of models with an isotropic cosmological initial singularity is approximated by the flat model corresponding to the 'equilibrium state' $\mathcal{F}_b$, characterized by $\Omega_b = 1$, $\mathbf{X} = \mathbf{0}$, which corresponds to a self-similar, spatially homogeneous and isotropic non-general-relativistic brane-world model [4]. This is consistent with previous work in which it was shown that for all physically relevant values of γ, $\mathcal{F}_b$ is a past-attractor in non-tilting spatially homogeneous brane-world models [9], and $\mathcal{F}_b$ is a past-attractor in the family of spatially inhomogeneous 'non-tilting' G_2 cosmological models [9]. The results are also consistent with previous results in the spatially homogeneous orthogonally transitive *tilting* Bianchi type VI_0 and VII_0 models [24], and Bianchi type VI_0 models with magnetic field [25] (note, especially, the bifurcation at 4/3). It follows immediately that the total energy density $\tilde{\rho} \to \infty$ as $t \to -\infty$ [9], so that $\mu_b \sim {\rho_b}^2$ dominates as $t \to -\infty$ and that all of the other contributions to the brane energy density are negligible dynamically as the singularity is approached. Hence close to the singularity the matter contribution is given by

$$\rho^{\rm tot} = \frac{1}{2\lambda}\rho^2 \equiv \mu_b; \quad p^{\rm tot} = \frac{1}{2\lambda}(\rho^2 + 2\rho p) = (2\gamma - 1)\rho^{\rm tot}, \tag{6}$$

so that the effective equation of state at high densities is $(2\gamma - 1)$, which is greater than unity in cases of physical interest.

More detailed information has recently been obtained from a study of the dynamics of a class of *spatially inhomogeneous* G_2 cosmological models with one spatial degree of freedom in the brane-world scenario [15]. The G_2 cosmological models admit a 2-parameter Abelian isometry group acting orthogonally transitively on spacelike 2-surfaces. The formalism of [16] was utilized, in which area expansion normalized scale-invariant dependent variables (the area expansion rate is effectively the Hubble rate close to an initial singularity), the timelike area gauge and an effective logarithmic proper time t were employed, and the initial singularity occurs for $t \to -\infty$. The resulting governing system of evolution equations of the spatially inhomogeneous G_2 brane cosmological models is then written as a system of autonomous first-order partial differential equations in two independent variables. In terms of scale-invariant dependent variables, the equations consist of an *Evolution system*: $\partial_t\{\mathbf{X}, \Omega_b\} = \mathbf{F}(\mathbf{X}, \Omega_b, \partial_x\mathbf{X}, \partial_x\Omega_b; \gamma)$, and *Constraint equations and defining equations*: $\mathbf{G}(\mathbf{X}, \Omega_b, \partial_x\mathbf{X}, \partial_x\Omega_b; \gamma) = 0$ [15]. The asymptotic evolution of the class of orthogonally transitive G_2 cosmologies near the cosmological initial singularity can then be discussed. Since the normalized frame variable was found to vanish asymptotically, the singularity is characterized by the fact that spatial derivatives are dynamically negligible.

The local dynamical behaviour of this class of spatially inhomogeneous models close to the singularity was then studied *numerically* [15]. It was found that the area expansion rate increases without bound (and hence the Hubble rate $H \to \infty$) as $t \to -\infty$, so that there always exists an initial singularity. For $\gamma > 4/3$, the numerics indicate $\{\mathbf{X}\} \to \mathbf{0}$ as $t \to -\infty$ (and $\Omega_b \to 1$) for *all* initial conditions. In the case of radiation ($\gamma = 4/3$), the models were still found to isotropize as $t \to -\infty$, albeit slowly. For $\gamma < 4/3$, $\{\mathbf{X}\}$ tend to constant but not necessarily zero values as $t \to -\infty$. In fact, the numerical results support the fact that *all* cosmological models have an isotropic singularity for $\gamma > 4/3$ (i.e., the singularity is isotropic for all initial conditions, indicating that $\mathcal{F}_b$) is a global past-attractor).

From the numerical analysis we conclude that there is an initial isotropic singularity in all of these G_2 spatially inhomogeneous brane cosmologies for a range of parameter values which include the physically important cases of radiation and a scalar field source. The numerical results are supported by a qualitative dynamical analysis and a detailed calculation of the past asymptotic decay rates. Although the analysis is local in nature, the numerics indicates that the singularity is isotropic for all initial conditions for the range of parameter values of physical import [15].

These results have been further supported by a detailed analysis of *linear perturbations* of the isotropic brane model $\mathcal{F}_b$ using the covariant and gauge invariant approach [19]. In particular, a detailed analysis of generic linear inhomogeneous and anisotropic perturbations [18] of the past attractor $\mathcal{F}_b$ was carried out by deriving a full set of linear 1+3 covariant *propagation* and *constraint* equations for this background, split into scalar, vector and tensor parts, which govern the complete perturbation dynamics of the physical quantities that describe the kinematics of the fluid flow and the dynamics of the gravitational field. The analysis was restricted to large scales, at a time when physical perturbation scales are much larger than the Hubble radius, $\lambda \gg H^{-1}$, which is of relevance for the discussion of non-inflationary perfect fluid models. In fact, since any wavelength $\lambda < H^{-1}$ at a given time becomes much larger than H^{-1} at earlier times, this perturbation analysis is completely general. Solutions to the set of perturbation equations were presented in Table I in [19], where it was concluded that $\mathcal{F}_b$ is stable in the past to generic inhomogeneous and anisotropic perturbations for physically relevant values of γ. In addition, it follows immediately that the expansion normalised shear vanishes as as the initial singularity is approached, and isotropization occurs.

We have argued that in spatially inhomogeneous brane-world cosmological models the initial singularity is *isotropic* [11]. Therefore, unlike the situation in GR, it is plausible that typically the initial singularity is isotropic in brane world cosmology. Such a 'quiescent' cosmology [20], in which the universe began in a highly regular state but subsequently evolved towards irregularity, might offer an explanation of why our Universe might have began its evolution in such a smooth manner and may provide a realisation of Penrose's ideas on gravitational entropy and the second law of thermodynamics in cosmology [21]. More importantly, it is therefore possible that a quiescent cosmological period occuring in brane cosmology provides a physical scenario in which the universe starts off smooth and that naturally gives rise to the conditions for inflation to subsequently take place. Cosmological observations indicate that we live in a Universe which is remarkably uniform on very large scales. However, the spatial homogeneity and isotropy of the Universe is difficult to explain within the standard GR framework since, in the presence of matter, the class of solutions to the Einstein equations which evolve towards a RW universe is essentially a set of measure zero. In the inflationary scenario, we live in an isotropic region of a potentially highly irregular universe as the result of an expansion phase in the early universe thereby solving many of the problems of cosmology. Thus this scenario can successfully generate a homogeneous and isotropic RW-like universe from initial conditions which, in the absence of inflation, would have resulted in a universe far removed from the one we live in today. However, still only a restricted set of initial data will lead to smooth enough conditions for the onset of inflation.

Let us discuss this in a little more detail. Although inflation gives a natural solution of the horizon problem of the big-bang universe, inflation requires homogeneous initial conditions over the super-horizon scale, i.e., it itself requires certain improbable initial conditions. When inflation begins to act, the universe must already be smooth on a scale of at least 10^5 times the Planck scale. Therefore, we cannot say that it is a solution of the horizon problem, though it reduces the problem by many orders of magnitude. Many people have investigated how initial inhomogeneity affects

the onset of inflation, and it was found that including spatial inhomogeneities accentuates the difference between models like new inflation and those like chaotic inflation. Goldwirth and Piran [22], who solved the full Einstein equations for a spherically symmetric spacetime, found that *small-field* inflation models of the type of *new inflation* is so sensitive to initial inhomogeneity that it requires homogeneity over a region of several horizon sizes. *Large-field* inflation models such as *chaotic inflation* is not so affected by initial inhomogeneity but requires a sufficiently high average value of the scalar field over a region of several horizon sizes [23]. Therefore, inhomogeneities further reduce the measure of initial conditions yielding new inflation, whereas the inhomogeneities have sufficient time to redshift in chaotic inflation, letting the zero mode of the field eventually drive successful inflation. In conclusion, although inflation is a possible causal mechanism for homogenization and isotropization, there is a fundemental problem in that the initial conditions must be sufficiently smooth in order for inflation to subsequently take place [10]. We have found that an isotropic singularity in brane world cosmology might provide for the necessary sufficiently smooth initial conditions to remedy this problem.

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